

EQUIVALENCE OF PAIRS OF BILINEAR OR QUADRATIC FORMS UNDER RATIONAL TRANSFORMATION*

BY

LEONARD EUGENE DICKSON

Introduction and Summary.

We consider pairs of bilinear or quadratic forms A and B with coefficients in a given field F . We seek necessary and sufficient conditions for the equivalence of A, B with A', B' under a linear transformation with coefficients in F . According as the determinant $|\lambda A + \mu B|$ is or is not identically zero, the case is called singular or non-singular, respectively. Both cases are treated in this paper.

In the non-singular case we may, without loss of generality (§ 3), assume that $|A| \neq 0, |A'| \neq 0$. By a well-known theorem, readily proved, a necessary condition for the equivalence in F of the two pairs of forms is that they shall have the same elementary divisors $(\lambda - c_i)^{e_i}$, or, if we prefer, the same rationally determined invariant factors. This condition is known to be sufficient when the forms are bilinear (§ 4), but is in general not sufficient when the forms are quadratic.

To obtain the advantage of decided simplification in the algebraic manipulation, we adjoin the roots c_i of $|\lambda A - B| = 0$ and operate in the field $F(c_1, \dots, c_n)$. The treatment is such that the irrationalities c_i may ultimately be eliminated and the operations replaced by rational operations in the initial field F . This is accomplished by employing new variables falling into sets of conjugates with respect to F (§ 1). Of several methods of treating bilinear forms in the field of all complex numbers, that by WEIERSTRASS is best adapted to the generalization to an arbitrary field F , but with a certain modification (§ 2) to secure the desired conjugacy of variables. To accomplish this essential simplification in the nature of the new variables, we must content ourselves with a preliminary normal form (2) involving certain factors f_σ not occurring in WEIERSTRASS's unique normal form. In view of the resulting conjugacies of the new variables with respect to F , two pairs of forms A, B and A', B' , having the same elementary divisors, are equivalent in F if and only if their normal forms (2) and

* Presented to the Society (Chicago) April 10, 1909.

(2') are equivalent in $F(c_1, \dots, c_n)$. This is always the case for bilinear forms (§ 4). For quadratic forms, the question of the equivalence of (2) and (2') is shown to reduce to the question of the equivalence of corresponding component pairs, where a component contains only the terms having equal c_σ and equal e_σ (and hence equal elementary divisors). Two component pairs corresponding to an elementary divisor of multiplicity s (i. e., occurring s times) are equivalent if and only if the corresponding s -ary quadratic forms $\sum f_\sigma z_\sigma^2$ and $\sum f'_\sigma z_\sigma^2$, obtained by* replacing each WEIERSTRASS combination Z by the square of a single variable z , are equivalent in $F(c_\sigma)$. Hence *the question of the equivalence of pairs of quadratic forms has been reduced to that for single † quadratic forms*. For the field of all real numbers or for any finite field, the criteria are quite simple (§§ 12, 13) and there results a complete classification of pairs of quadratic forms in these fields.

The success of the method of treatment depends not only upon the use of conjugate variables but also upon the employment of a certain device. It appears to be impracticable to consider the system of quadratic conditions which arise when we proceed directly to determine a linear transformation T which shall replace a given pair of quadratic forms A, B by a second given pair A', B' . We make use of the following indirect procedure. By the differentiation of $A' = A$ and of the equations for T , we obtain a necessary form for T^{-1} ; also a second form from $B' = B$. The conditions that the resulting two inverses of T shall be identical are linear in the unknown coefficients of T .

The same device is employed in § 14 to reduce the singular case to the non-singular case. We here make use of the rational normal type obtained by KRONECKER in his fundamental paper of 1890. This classic work of KRONECKER, exposing the very heart of the subject and applicable in complete generality to any domain of rationality, has not been in the least superseded by subsequent work, as claimed by MUTH.‡ It is not necessary to discredit the work of the master KRONECKER, writing in the spirit of general algebra, in order to honor FROBENIUS for his later elegant proof for the special case of the domain of all complex numbers.

Introduction of variables conjugate with respect to F .

1. Beginning with an example, let c_1, c_2, c_3 be the roots of a cubic equation irreducible in the field F . If x_1, x_2, x_3 are the initial variables, let

$$T: \quad z_i = \alpha(c_i)x_1 + \beta(c_i)x_2 + \gamma(c_i)x_3 \equiv X_1 + c_i X_2 + c_i^2 X_3 \quad (i=1, 2, 3)$$

be linearly independent functions in which the coefficients of the polynomials

* Or, by replacing e_σ by unity.

† These Transactions, vol. 7 (1906), p. 275; Bulletin of the American Mathematical Society, vol. 14 (1907), p. 108.

‡ *Elementartheiler*, bottom of p. 125, top of p. 126.

$\alpha(\rho), \dots$ belong to F . Then X_1, X_2, X_3 are independent linear functions of the x_i with coefficients in F . Let $Q(x)$ be a quadratic form, with coefficients in F , which is reducible to the form $N(z)$ by the transformation T . Let $q(y)$ be a second form reducible to $N(z)$ by the analogous transformation

$$T': \quad z_i = \lambda(c_i)y_1 + \mu(c_i)y_2 + \nu(c_i)y_3 \equiv Y_1 + c_i Y_2 + c_i^2 Y_3 \quad (i=1, 2, 3)$$

Then $Q(x)$ is reduced to $q(y)$ by the transformation $x_i = \sum d_{ij}y_j$ obtained by solving $X_i = Y_i$ and hence with coefficients d in F .

For either T or T' the variables z_i are said to be conjugate with respect to the field F , since z_i is derived from z_1 by replacing c_1 by a root c_i of the same irreducible equation.

In general, let x_1, \dots, x_n be the initial variables. For $n = d_1 + \dots + d_r$, let $D_s(\lambda)(s=1, \dots, r)$ be irreducible in F and of degree d_s ; the case in which two or more D 's coincide is not excluded. Let $c_{si}(i=0, \dots, d_s-1)$ be the roots of $D_s=0$. We introduce r sets of new variables, those in the s th set being

$$z_{si} = \sum_{k=1}^n \alpha_{sk}(c_{si})x_k \equiv \sum_{j=0}^{d_s-1} f_{sj}(x)c_{si}^j \quad (i=0, \dots, d_s-1),$$

in which the coefficients of the functions α and f belong to the field F . Since z_{si} is obtained from z_{s0} by replacing c_{s0} by c_{si} , the d_s new variables z_{si} are said to be conjugate with respect to F . Now the determinant

$$|c_{si}^j| \neq 0 \quad (i, j=0, \dots, d_s-1).$$

Hence, for each s , $f_{sj}(j=0, \dots, d_s-1)$ may be introduced as new variables in place of $z_{si}(i=0, \dots, d_s-1)$. Thus the r sets of conjugate variables z_{si} may be replaced by n independent variables $f_{sj}(x)$ with coefficients in F .

If in the normal forms in the z 's for two bilinear or quadratic forms A and B , we replace the z_{si} by the f_{sj} , we obtain normal forms with coefficients in F which define a decomposition of A and B into component pairs each indecomposable in F (§ 11).

The modified WEIERSTRASS normal form with conjugate variables, §§ 2-3.

2. We make use of WEIERSTRASS's* reduction of a pair of bilinear or quadratic forms, but with a certain modification which leads to new variables falling into sets of conjugates with respect to the initial field F . The latter is any field, not having modulus 2, which contains the coefficients of the forms. While the introduction of the roots c_σ of the characteristic equation $|\lambda A - B| = 0$ is essential to the theory, this is not the case with the further irrationalities $\sqrt{C_\kappa}$. By avoiding the introduction of the latter, we shall secure

* We shall refer to the exposition by MUTH, *Elementarteiler*, pp. 69-85, 118-1:2, which embodies certain results by FROBENIUS essential to the completion of WEIERSTRASS's treatment.

the desired conjugacies of the new variables with respect to F . To this end we modify WEIERSTRASS's procedure (MUTH, p. 73), as follows: We set

$$S^{(\kappa-1)}S^{(\kappa)} = (\lambda - c)^{l_{\kappa-1}+l_{\kappa}} C_{\kappa} T,$$

where T is a rational integral function of $\lambda - c$ with the constant term unity. Hence T equals the square of a power series q in $\lambda - c$ with the constant term unity. Proceeding as in MUTH, p. 74, with Q replaced by q , we obtain his formula (11) with the irrational factor deleted, and his formula (12) with the additional factor $f_{\kappa} = C_{\kappa}^{-1}$ inserted. Ultimately we obtain the following result:

If A and B are bilinear forms on $2n$ variables with coefficients in the field F , and $|A| \neq 0$, and if the elementary divisors of $|\lambda A - B|$ are

$$(1) \quad (\lambda - c_1)^{e_1}, \dots, (\lambda - c_m)^{e_m} \quad (e_1 + \dots + e_m = n),$$

there exist $2n$ independent linear homogeneous functions $X_{\sigma\mu}$, $Y_{\sigma\nu}$ of the initial variables such that f_{σ} and the coefficients of $X_{\sigma\mu}$, $Y_{\sigma\nu}$ are rational integral functions of c_{σ} in F , and such that

$$(2) \quad A = \sum_{\sigma=1}^m f_{\sigma} Z_{e_{\sigma}}, \quad B = \sum_{\sigma=1}^m f_{\sigma} (c_{\sigma} Z_{e_{\sigma}} + Z_{e_{\sigma-1}}), \quad Z_{e_{\sigma}} \equiv \sum_{\mu=0}^{e_{\sigma}-1} X_{\sigma\mu} Y_{\sigma e_{\sigma-1}-\mu},$$

where the second Z in B is to be deleted if $e_{\sigma} = 1$. If A and B are quadratic forms, $Y_{\sigma\mu} = X_{\sigma\mu}$.

If c_1, \dots, c_r are the roots of the same irreducible factor in F of $|\lambda A - B|$, then $X_{1\mu}, \dots, X_{r\mu}$ are conjugate with respect to F ; likewise $Y_{1\nu}, \dots, Y_{r\nu}$. We have therefore secured the desired conjugacies of the new variables.

3. If we remove the restriction that $|A| \neq 0$ and assume merely that the determinant $|\lambda A + \mu B|$ is not identically zero in λ, μ , we readily obtain a pair of forms \bar{A}, \bar{B} to which the preceding theorem may be applied. Let g and h be constants in F for which $|gA + hB| \neq 0$, g' and h' constants in F for which $gh' - g'h = 1$. Then (MUTH, p. 83),

$$\bar{A} = gA + hB, \quad \bar{B} = g'A + h'B$$

can be reduced to the type (2). We may then deduce normal forms for A, B . Instead of the latter more complicated forms, it suffices to employ (2) in determining the conditions for the equivalence of A, B with A', B' in the field F . For, if these pairs are equivalent in F , then \bar{A}, \bar{B} and

$$\bar{A}' = gA' + hB', \quad \bar{B}' = g'A' + h'B'$$

are equivalent in F , and conversely.

4. For the case of bilinear forms we set

$$X_{\sigma\mu} = f_{\sigma}^{-1} X_{\sigma\mu}^*, \quad Y_{\sigma\mu} = Y_{\sigma\mu}^* \quad (\sigma = 1, \dots, m; \mu = 0, \dots, e_{\sigma} - 1),$$

and obtain from (2) a normal form in which each $f_{\sigma}^* = 1$. The new variables X^* , Y^* fall into sets of conjugates with respect to F' . Hence in the non-singular case, two pairs of bilinear forms with coefficients in the field F' are equivalent in F' if and only if their characteristic determinants have the same elementary divisors (or the same invariant factors). We have therefore obtained by a modification of WEIERSTRASS'S method a proof of the theorem due to FROBENIUS.

A like argument does not hold for quadratic forms.

Ultimate normal forms, §§ 5-10.

5. Henceforth, we consider the equivalence of a pair of quadratic forms (2), for $Y = X$, with a second pair

$$(2') \quad A' = \sum_{\sigma=1}^m f'_{\sigma} Z'_{\sigma}, \quad B' = \sum_{\sigma=1}^m f'_{\sigma} (c_{\sigma} Z'_{\sigma} + Z'_{e_{\sigma}-1}), \quad Z'_{e_{\sigma}} \equiv \sum_{\mu=0}^{e_{\sigma}-1} X'_{\sigma\mu} X'_{\sigma, e_{\sigma}-1-\mu},$$

having the same elementary divisors as the pair (2). Here f'_{σ} , like f_{σ} , is a non-vanishing rational integral function of c_{σ} with coefficients in F' . The variables X are conjugate linear functions of the initial variables x_i , and the X' are conjugate functions of the x'_i . Hence (§ 1) the initial pairs A , B and A' , B' are equivalent in F' if and only if their normal forms (2) and (2') are equivalent under a transformation

$$(3) \quad X_{\sigma\mu} = \sum_{s=1}^m \sum_{t=0}^{e_s-1} d_{st}^{\sigma\mu} X'_{st} \quad (\sigma = 1, \dots, m; \mu = 0, \dots, e_{\sigma}-1),$$

in which the d 's are elements of $F'(c_1, \dots, c_m)$ which satisfy certain conditions imposed by the conjugacies of the variables.

We seek the conditions under which $A' = A$, $B' = B$ in view of relations (3). By differentiation with respect to X'_{st} , we get

$$(4) \quad \begin{aligned} 2f'_s X'_{s, e_s-1-t} &= \sum_{\sigma=1}^m f'_{\sigma} K_{\sigma\sigma}, \\ 2f'_s c_s X'_{s, e_s-1-t} + 2f'_s X'_{s, e_s-2-t} &= \sum_{\sigma=1}^m f'_{\sigma} (c_{\sigma} K_{\sigma\sigma} + K_{\sigma, e_{\sigma}-1}), \end{aligned}$$

where the second X' in (4)₂ is to be suppressed if $t = e_s - 1$, while

$$K_{\sigma\sigma} \equiv \sum_{\mu=0}^{e_{\sigma}-1} X_{\sigma\mu} d_{st}^{\sigma\mu'} + \sum_{\mu=0}^{e_{\sigma}-1} X_{\sigma\mu'} d_{st}^{\sigma\mu} \quad (\mu' = e_{\sigma} - 1 - \mu).$$

Since μ and μ' range over the same values, the sums are equal and

$$(5) \quad K_{\sigma\sigma} = 2 \sum_{\mu=0}^{e_{\sigma}-1} d_{st}^{\sigma\mu} X_{\sigma, e_{\sigma}-1-\mu}.$$

Hence (4₁) and (4₂) must define the same transformation, viz., the inverse of (3). Conversely,* if (4₁) and (4₂) define the same transformation inverse to (3) and if $|d| \neq 0$, then the pair (2) is equivalent to the pair (2') under the transformation (3).

6. Replacing e_σ by $e_\sigma - 1$ and μ by $\mu - 1$ in (5), we get

$$(6) \quad K_{\sigma, \sigma-1} = 2 \sum_{\mu=1}^{e_\sigma-1} d_{\sigma, t}^{\sigma, \mu-1} X_{\sigma, e_\sigma-1-\mu}.$$

We compare the relations (4) with the values (5) and (6) inserted. For $t = e_\sigma - 1$, (4₂) must equal the product of (4₁) by c_σ , whence

$$(c_\sigma - c_\sigma) d_{\sigma, e_\sigma-1}^{\sigma, 0} = 0, \quad (c_\sigma - c_\sigma) d_{\sigma, e_\sigma-1}^{\sigma, \mu} = d_{\sigma, e_\sigma-1}^{\sigma, \mu-1} \quad (\sigma = 1, \dots, m; \mu = 1, \dots, e_\sigma - 1).$$

It follows readily that

$$(7) \quad d_{\sigma, e_\sigma-1}^{\sigma, \mu} = 0 (0 \leq \mu \leq e_\sigma - 1) \text{ for } c_\sigma \neq c_\sigma; \quad d_{\sigma, e_\sigma-1}^{\sigma, \nu} = 0 (0 \leq \nu \leq e_\sigma - 2) \text{ for } c_\sigma = c_\sigma.$$

For $0 \leq t \leq e_\sigma - 2$, we substitute into (4₂) the expression (4₁) for X' and that derived from it by replacing t by $t + 1$. We obtain the conditions

$$(8) \quad (c_\sigma - c_\sigma) d_{\sigma, t}^{\sigma, 0} = d_{\sigma, t+1}^{\sigma, 0}, \quad (c_\sigma - c_\sigma) d_{\sigma, t}^{\sigma, \mu} = d_{\sigma, t+1}^{\sigma, \mu} - d_{\sigma, t}^{\sigma, \mu-1} \quad (\mu = 1, \dots, e_\sigma - 1).$$

First, let σ and s be such that $c_\sigma \neq c_s$. For $\mu = 0$ in (7₁) and $t = e_\sigma - 2$ in (8₁), we get $d_{\sigma, e_\sigma-2}^{\sigma, 0} = 0$. Then for $t = e_\sigma - 3$ in (8₁), we get $d_{\sigma, e_\sigma-3}^{\sigma, 0} = 0$. Proceeding similarly with (8₁), we establish (9) for $i = 0$:

$$(9) \quad d_{\sigma, k}^{\sigma, i} = 0 \quad (0 \leq i \leq e_\sigma - 1, 0 \leq k \leq e_\sigma - 1, c_\sigma \neq c_s).$$

To proceed by induction, let (9) hold for $i = \mu - 1$. Then the final term in (8₂) is zero. Take $t = e_\sigma - 2$ in (7₁); we get $d_{\sigma, e_\sigma-2}^{\sigma, \mu} = 0$. For $t = e_\sigma - 3$ in (8₂), we get $d_{\sigma, e_\sigma-3}^{\sigma, \mu} = 0$. Proceeding similarly with (8₂), we find that (9) holds for $i = \mu$. Hence the induction is complete.

Next, let $c_\sigma = c_s$. Then (7₂) and (8) are satisfied if and only if

$$(10) \quad d_{\sigma, k}^{\sigma, i} = 0 \text{ for } k > i, \quad d_{\sigma, k}^{\sigma, i} = d_{\sigma, 0}^{\sigma, i-k} \text{ for } k \leq i \quad (0 \leq i \leq e_\sigma - 1, 0 \leq k \leq e_\sigma - 1, c_\sigma = c_s).$$

7. If we separate the variables X into sets, including in a single set all the $X_{\sigma, \mu}$ for which the c_σ are equal, we see from (9) that the variables of any set are transformed amongst themselves by (3). We may therefore limit attention to the separate sets. Consider the set for which

$$(11) \quad c_1 = c_2 = \dots = c_g, \quad c_\gamma \neq c_1 (\gamma > g).$$

* For a formal verification, multiply (4₁) $X_{\sigma, t}$ and sum for $s = 1, \dots, m$; $t = 0, \dots, e_\sigma - 1$. Applying (2₁) on the left, and (3) and (2₂) on the right, we get $A' = A$. Proceeding similarly with (4₂), we get $B' = B$.

By (9) and (10), the corresponding partial transformation (3) is

$$(12) \quad X_{\sigma\mu} = \sum_{s=1}^g \sum_{t=0}^{l_s} d_{s0}^{\sigma\mu-t} X'_{st} \quad (\sigma=1, \dots, g; \mu=0, \dots, e_\sigma-1),$$

where l_s denotes the lesser of the integers $\mu, e_s - 1$. By (4₁) and (5),

$$f'_s X'_{s, e_s-1-t} = \sum_{\sigma=1}^g f_\sigma \sum_{\mu=t}^{e_\sigma-1} d_{s0}^{\sigma\mu-t} X_{\sigma, e_\sigma-1-\mu} \quad (s=1, \dots, g; t=0, \dots, e_s-1).$$

By a change of indices, the latter becomes

$$(13) \quad f'_s X'_{st} = \sum_{\sigma=1}^g f_\sigma \sum_{\mu=0}^{t+e_\sigma-e_s} d_{s0}^{\sigma\mu-e_s+t-\mu} X_{\sigma\mu} \quad (s=1, \dots, g; t=0, \dots, e_s-1).$$

The partial transformation (12) must replace the terms of (2) with $\sigma \leq g$ by the terms of (2') with $\sigma \leq g$. Subtracting c_1 times (2₁) from (2₂), and c_1 times (2'₁) from (2'₂), we find by (11) that

$$(14) \quad \sum_{\sigma=1}^g f_\sigma Z_{e_\sigma} = \sum_{\sigma=1}^g f'_\sigma Z'_{e_\sigma}, \quad \sum_{\sigma=1}^g f_\sigma Z_{e_\sigma-1} = \sum_{\sigma=1}^g f'_\sigma Z'_{e_\sigma-1}$$

under transformation (12). The necessary and sufficient condition for the equivalence of the pairs (14) under (12) is that transformation (13) be inverse to (12) (see end of § 5). Since (14) are unaltered by the interchange of the accented and unaccented letters and since (13) must replace the second pair by the first, it follows that (13) must be of the same type as (12), the most general transformation replacing the first pair by the second. Hence

$$(15) \quad d_{s0}^{\sigma\mu-e_s+t-\mu} = 0 \quad (\mu=t+1, \dots, t+e_\sigma-e_s, \text{ if } e_\sigma > e_s),$$

$$(15') \quad d_{s0}^{\sigma k} = 0 \quad (k=0, \dots, e_\sigma-e_s-1, \text{ if } e_\sigma > e_s).$$

These equivalent formulæ are fundamental in what follows.

8. We proceed to determine and discuss the conditions under which (12) and (13) are inverse transformations. Eliminating* $X_{\sigma\mu}$ and changing† the order of the summations, we get

$$(16) \quad f'_s X'_{st} = \sum_{s_1=1}^g \sum_{t_1=0}^{\tau} \phi_{s_1 t_1} X'_{s_1 t_1}, \quad \phi_{s_1 t_1} = \sum_{\sigma=1}^g \sum_{\mu=t_1}^{\tau+e_\sigma-e_s} f_\sigma d_{s0}^{\sigma\mu-e_s+t-\mu} d_{s_1 0}^{\sigma\mu-t_1},$$

where τ is the lesser of $t, e_{s_1}-1$. Hence $\phi = 0$ except for $s_1 = s, t_1 = t$.

We employ the abbreviation (s) to denote the set of all integers s corresponding to equal values of e_s , so that s_1 is in (s) if and only if $e_{s_1} = e_s$.

We may give a very simple expression for $\phi_{s_1 t}$ when s_1 is in the set (s) .

In (13), $\mu \leq t+e_\sigma-e_s \leq e_\sigma-1$, so that each $X_{\sigma\mu}$ may be replaced by (12).

† By (15), the maximum μ in (13) is t .

Consider first the values of σ for which $e_\sigma > e_s$. Then, in (16₂) with $t_1 = t$, the first d vanishes by (15) when $\mu > t$, while the second d vanishes by (15') when $\mu = t$. For $e_\sigma < e_s$, there occur no terms in the sum. Hence we may restrict σ to the set (s) , so that

$$(17) \quad \phi_{s_1 t} \equiv \sum_{\sigma \text{ in } (s)} f_\sigma d_{s_0}^{\sigma_0} d_{s_1 0}^{\sigma_1} = \begin{cases} f'_s & \text{if } s_1 = s \\ 0 & \text{if } s_1 \neq s \end{cases} \quad [s_1 \text{ in } (s)]$$

Now (17) are the necessary and sufficient conditions that

$$(18) \quad z_\sigma = \sum_{s_1 \text{ in } (s)} d_{s_1 0}^{\sigma_0} z'_{s_1} \quad [\sigma \text{ in } (s)]$$

shall replace

$$(19) \quad \sum f_\sigma z_\sigma^2 \quad \text{by} \quad \sum f'_\sigma z_\sigma'^2 \quad [\sigma \text{ in } (s)]$$

Since $\Pi f'_\sigma = |d|^2 \Pi f_\sigma$ and each $f'_\sigma \neq 0$, $f_\sigma \neq 0$, we have*

$$(20) \quad |d_{s_0 0}^{\sigma_0}| \neq 0 \quad [\sigma, s_1 \text{ in } (s)]$$

For the existence in $F(c_1)$ of elements $d_{s_0}^{\sigma_0}$ making each $\phi_{s_1 t_1} = 0$ or f'_s , a necessary condition was seen to be the existence of elements

$$(21) \quad d_{s_0}^{\sigma_0} \quad [\sigma \text{ in } (s); s = 1, \dots, g]$$

which satisfy relations (17). This condition is also sufficient. In proof, we assign the value zero to every d not in the set (21). Then $\phi_{s_1 t_1}$ vanishes unless σ and s_1 are in (s) and $t = \mu = t_1$, while in that case ϕ becomes (17). Hence (12) and (13) are then inverse, a fact also evident from their special forms:

$$(22) \quad X_{\sigma\mu} = \sum_{s \text{ in } (\sigma)} d_{s_0}^{\sigma_0} X'_{s\mu}, \quad f'_s X'_{st} = \sum_{\sigma \text{ in } (s)} f_\sigma d_{s_0}^{\sigma_0} X_{\sigma t}.$$

For each set (s) , relations (17) express the equivalence of the quadratic forms (19) under a transformation in $F(c_1)$ of non-vanishing determinant.

9. The discussion in §§ 5–8 leads to the following

THEOREM. *Let the elementary divisors $(\lambda - c_i)^{e_i}$ be separated into sets by listing together all the equal ones (with equal c 's and equal e 's). Let the first set be given by $i = 1, \dots, \gamma_1$; the second by $i = \gamma_1 + 1, \dots, \gamma_1 + \gamma_2$; the third by $\gamma_1 + \gamma_2 + 1, \dots, \gamma_1 + \gamma_2 + \gamma_3$, etc. Then the necessary and sufficient condition for the equivalence in the field $F(c_1, \dots, c_m)$ of the two pairs of*

* Another proof follows from (12) for $\mu = 0$ and (15') for $k = 0$. Let e_{σ_1} be the greatest e , e_{σ_2} the next to the greatest e , etc. Then

$$X_{\sigma_1 0} = \sum_{s \text{ in } (\sigma_1)} d_{s_0}^{\sigma_1 0} X'_{s_0 0}, \quad X_{\sigma_2 0} = \sum_{s \text{ in } (\sigma_1)} d_{s_0}^{\sigma_2 0} X'_{s_0 0} + \sum_{s \text{ in } (\sigma_2)} d_{s_0}^{\sigma_2 0} X'_{s_0 0}, \dots$$

Hence the determinant of (12) contains the non-vanishing factors

$$|d_{s_0}^{\sigma_1 0}|, s \text{ and } \sigma_1 \text{ in } (\sigma_1); \quad |d_{s_0}^{\sigma_2 0}|, s \text{ and } \sigma_2 \text{ in } (\sigma_2); \dots$$

quadratic forms (2) and (2') is the equivalence in $F(c_\sigma)$ of each individual quadratic form

$$(23) \quad \sum_{\sigma=1}^{\gamma_1} f_\sigma z_\sigma^2, \quad \sum_{\sigma=\gamma_1+1}^{\gamma_1+\gamma_2} f_\sigma z_\sigma^2, \quad \sum_{\sigma=\gamma_1+\gamma_2+1}^{\gamma_1+\gamma_2+\gamma_3} f_\sigma z_\sigma^2, \dots$$

with the corresponding form $\sum f'_\sigma z_\sigma^2$. The number of forms (23) is the number of distinct elementary divisors.

If c_1, \dots, c_r are the roots of an irreducible factor in F of the characteristic function $|\lambda A - B|$, the relation between the elementary divisors and the rational invariant factors shows that the elementary divisors which correspond to $\lambda - c_i$ ($i = 2, \dots, r$) may be derived from those which correspond to $\lambda - c_1$ by replacing c_1 by c_i . Hence the equivalence in $F(c_1)$ of those of the forms (23) which relate to c_1 with the corresponding forms (23') implies the equivalence in $F(c_i)$ of the forms relating to c_i .

10. Another convenient statement of the theorem of § 9 is as follows. The equivalence of the pairs (2) and (2') requires the equivalence of the component pairs obtained by limiting the summation index σ to values giving equal c_σ 's. If one set of equal c 's is given by (11), the pairs (14) must be equivalent. The latter requires the equivalence of the component pairs obtained by the further limitation of σ to a set (σ) of values giving equal e 's. Hence for each set (σ) ,

$$(24) \quad \sum_{(\sigma)} f_\sigma Z_{e_\sigma}, \quad \sum_{(\sigma)} f'_\sigma Z_{e_{\sigma-1}}$$

must be equivalent to the corresponding forms (24') in f'_σ . Here σ ranges over the integers (σ) for which the c_σ are equal and the e_σ are equal. Thus the pairs (2) and (2') are equivalent in $F(c_1, \dots, c_m)$ if and only if the component pairs (24) and (24') are equivalent in $F(c_\sigma)$. The latter pairs are equivalent if and only if the individual forms $\sum_{(\sigma)} f_\sigma z_\sigma^2$ and $\sum_{(\sigma)} f'_\sigma z_\sigma^2$ are equivalent in $F(c_\sigma)$. It thus suffices to replace * the combinations

$$(25) \quad Z_{e_\sigma} \equiv \sum_{\mu=0}^{e_\sigma-1} X_{\sigma\mu} X_{\sigma e_\sigma-1-\mu}$$

by the squares $z_{e_\sigma}^2$ of single new variables.

It will prove instructive to have a direct verification of the sufficiency of these conditions. By hypothesis the forms (19) are equivalent, so that relations (17)

*Or, to replace e_σ by unity.

hold. Now transformation (22₁) replaces the form (24₁) by

$$\sum_{\mu=0}^{e_{\sigma}-1} \sum_{s, s_1 \text{ in } (\sigma)} (\sum_{(\sigma)} f_{\sigma} d_{s_0}^{\sigma_0} d_{s_1}^{\sigma_1}) X'_{s\mu} X'_{s_1 e_{\sigma}-1-\mu}.$$

By (17), the quantity in parenthesis equals f'_{σ} or 0 according as $s_1 = s$ or $s_1 \neq s$. In view of (25) and $e_s = e_{\sigma}$, we obtain $\sum_{s \text{ in } (\sigma)} f'_{\sigma} Z'_{s_1}$. Similarly, (24₂) is transformed into (24'₂).

Decomposition of a pair of quadratic forms in F .

11. Let c_1, \dots, c_r be the distinct roots of an irreducible equation $\phi = 0$ in F . Of the elementary divisors which are powers of $\lambda - c_1$, consider a particular one $(\lambda - c_1)^{\epsilon_1}$ and the related variables

$$X_{1\mu} = V_{0\mu} + V_{1\mu} c_1 + \dots + V_{r-1\mu} c_1^{\epsilon_1-1} \quad (\mu = 0, \dots, \epsilon_1 - 1),$$

where the V 's are linear functions of the initial variables with coefficients in F . If we replace c_1 by c_{σ} ($1 \leq \sigma \leq r$), we obtain (end of § 9) an elementary divisor $(\lambda - c_{\sigma})^{\epsilon_1}$ and related variables $X_{\sigma\mu}$. Since $|c_{\sigma}^k| \neq 0$, the $V_{i\mu}$ may be expressed in terms of $X_{1\mu}, \dots, X_{r\mu}$. Hence the $r\epsilon_1$ functions $V_{i\mu}$ are linearly independent. Thus

$$\alpha(V) = \sum_{\sigma=1}^r f_{\sigma} Z_{c_{\sigma}}, \quad \beta(V) = \sum_{\sigma=1}^r f_{\sigma} (c_{\sigma} Z_{c_{\sigma}} + Z_{c_{\sigma}-1})$$

are quadratic forms on the independent variables $V_{i\mu}$ with coefficients in F .

The pair α, β has the single invariant factor ϕ^{ϵ_1} , and the elementary divisors $(\lambda - c_i)^{\epsilon_1}$ ($i = 1, \dots, r$). But if $\alpha = \alpha' + \alpha'', \beta = \beta' + \beta''$, the variables in α', β' being distinct from those in α'', β'' , the elementary divisors of α, β are those of α', β' , together with those of α'', β'' . Hence there is no such decomposition of α, β within the field F .

THEOREM. *If $|A| \neq 0$ and $|\lambda A - B|$ has the invariant factors*

$$(26) \quad \phi_{11}^{\epsilon_{11}} \phi_{12}^{\epsilon_{12}} \phi_{13}^{\epsilon_{13}} \dots, \phi_{21}^{\epsilon_{21}} \phi_{22}^{\epsilon_{22}} \phi_{23}^{\epsilon_{23}} \dots, \dots$$

in which each ϕ_{ij} is irreducible in F and each $\epsilon_{ij} > 0$, the pair of quadratic forms A, B decomposes in F into $\sum A_{ij}, \sum B_{ij}$, where each component pair A_{ij}, B_{ij} is not decomposable in F and has the single invariant factor $\phi_{ij}^{\epsilon_{ij}}$.

Field of reals; finite fields.

12. Let F be the field R of all real numbers. Consider first a pair of conjugate imaginary roots c_1 and c_2 , whence $e_1 = e_2$. In the field $C = F(c_1)$ of all complex numbers, $2f_1$ is a square g_1^2 and $2f_2$ is the conjugate imaginary g_2^2 . Set $X'_{1\mu} = g_1 X_{1\mu}$, $X'_{2\mu} = g_2 X_{2\mu}$. Hence we may set $f_1 = f_2 = \frac{1}{2}$ and yet preserve the conjugacy of the variables. With this preliminary normalization accomplished we set

$$X_{1\mu} = x_{1\mu} + iy_{1\mu}, \quad X_{2\mu} = x_{1\mu} - iy_{1\mu} \quad (\mu=0, \dots, e_1-1),$$

$$c_1 = \gamma_1 + i\delta_1, \quad c_2 = \gamma_1 - i\delta_1.$$

Then, by (25), $Z_{e_1} = L_{e_1} + iM_{e_1}$, where

$$(27) \quad L_{e_1} = \sum_{\mu=0}^{e_1-1} (x_{1\mu}x_{1e_1-1-\mu} - y_{1\mu}y_{1e_1-1-\mu}), \quad M_{e_1} = 2 \sum_{\mu=0}^{e_1-1} x_{1\mu}y_{1e_1-1-\mu}.$$

Let A_{12} and B_{12} denote the terms of (2) given by $\sigma = 1, 2$. Since

$$\frac{1}{2}(Z_{e_1} + Z_{e_2}) = L_{e_1}, \quad \frac{1}{2}(c_1 Z_{e_1} + c_2 Z_{e_2}) = \gamma_1 L_{e_1} - \delta_1 M_{e_1},$$

we have

$$(28) \quad A_{12} = L_{e_1}, \quad B_{12} = \gamma_1 L_{e_1} - \delta_1 M_{e_1} + L_{e_1-1}.$$

When the variables are taken in the order $x_{10}, y_{10}, x_{11}, y_{11}, \dots$, we have

$$(29) \quad (\lambda A_{12} - B_{12}) = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 0 & P & Q \\ 0 & 0 & 0 & \dots & 0 & P & Q & 0 \\ 0 & 0 & 0 & \dots & P & Q & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P & Q & 0 & \dots & 0 & 0 & 0 & 0 \\ Q & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{vmatrix}$$

where

$$(30) \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} \lambda - \gamma_1 & \delta_1 \\ \delta_1 & -\lambda + \gamma_1 \end{pmatrix}.$$

The determinant of (29) has the single invariant factor $|Q|^{e_1}$.

Next, let c_s be real and, among the elementary divisors which are powers of $\lambda - c_s$, let those exponents e be equal for which σ ranges over the set (s) . Since the corresponding f_σ and f'_σ are real, the two forms (19) are equivalent in R if and only if the number of positive f_σ equals the number of positive f'_σ , where σ ranges over (s) .

THEOREM. *The necessary and sufficient conditions for the equivalence under*

real linear transformation of two non-singular pairs of real quadratic forms are that they have the same elementary divisors and the same number of positive terms in the series $f_{\sigma_1}, \dots, f_{\sigma_s}$ and $f'_{\sigma_1}, \dots, f'_{\sigma_s}$ corresponding to each set of equal real elementary divisors $(\lambda - c_{\sigma_1})^{\sigma_1} \equiv \dots \equiv (\lambda - c_{\sigma_s})^{\sigma_s}$.

As a real normal form for a pair of real quadratic forms we may employ two aggregates of forms affecting different sets of variables. Corresponding to each pair of conjugate imaginary elementary divisors we take as the component forms a pair of type (28). Corresponding to each real elementary divisor of multiplicity s , we take as the component forms sums like (2), with σ ranging over s values, with the value $+1$ assigned to s' of the f_{σ} , $0 \leq s' \leq s$, and the value -1 assigned to the remaining $s - s'$ of the f_{σ} .

13. Let the field F be the $GF[p^r]$, $p > 2$. If c is the root of an equation of degree r irreducible in the $GF[p^r]$, the forms (19) are equivalent in $F(c)$, viz., in the $GF[p^{r'}]$, if and only if $\Pi f'_{\sigma} \div \Pi f_{\sigma}$ is a square in the $GF[p^{r'}]$.

To obtain normal forms for the non-equivalent pairs of quadratic forms in the $GF[p^r]$ we may therefore employ (2), with the f_{σ} chosen so that, corresponding to an elementary divisor of multiplicity s , $s - 1$ of the f_{σ} equal unity and the remaining one of the f_{σ} is unity or a particular not-square in the $GF[p^r]$.

Reduction of the singular case to the non-singular case.

14. KRONECKER* has established the following theorem: Within any field \dagger F not having modulus 2, a pair of quadratic forms ϕ, ψ , for which $|u\phi + v\psi| \equiv 0$, can be transformed into a pair

$$(31) \quad \begin{aligned} A &= \sum_{g,h}^{1, \dots, M} A_{gh} X_g X_h + \sum_{q=1}^L \sum_{p=1}^{M_q} X_{p-1}^{(q)} X_{p+M_q}^{(q)}, \\ B &= \sum B_{gh} X_g X_h + \sum \sum X_p^{(q)} X_{p+M_q}^{(q)}, \end{aligned}$$

in which $|uA_{gh} + vB_{gh}| \not\equiv 0$.

Let $\alpha, \beta, \gamma, \delta$ be elements of F such that

$$|\alpha A_{gh} + \beta B_{gh}| \not\equiv 0 \quad (\alpha\delta - \beta\gamma \not\equiv 0).$$

By a transformation on the variables $X^{(q)}$ alone, the pair

$$\alpha A + \beta B, \gamma A + \delta B$$

can be transformed into a pair A^*, B^* , of type \ddagger (31) with

* Berliner Sitzungsberichte, 1890, pp. 1375-1388. See p. 1388.

\dagger KRONECKER limits the statement of the theorem to a domain of rationality.

\ddagger KRONECKER, loc. cit., bottom of p. 1382.

$$A_{gh}^* = \alpha A_{gh} + \beta B_{gh}, \quad B_{gh}^* = \gamma A_{gh} + \delta B_{gh}.$$

We may therefore restrict attention to pairs (31) with $|A_{gh}| \neq 0$.

We investigate the equivalence of a pair (31) with a like pair

$$(31') \quad A' = \sum A'_{gh} Y_g Y_h + \sum \sum Y_{p-1}^{(q)} Y_{\mu}^{(q)}, \quad B' = \sum B'_{gh} Y_g Y_h + \sum \sum Y_p^{(q)} Y_{\mu}^{(q)} \\ (\mu = p + M_t)$$

under a linear transformation, with coefficients in F ,

$$(32) \quad X_h = \sum_{i=1}^M c_{hi} Y_i + \sum_{q=1}^L \sum_{j=0}^{2M_t} d_{hqj} Y_j^{(q)} \quad (h=1, \dots, M), \\ X_i^{(l)} = \sum_{i=1}^M e_{li} Y_i + \sum_{q=1}^L \sum_{j=0}^{2M_t} f_{liqj} Y_j^{(q)} \quad (l=1, \dots, L; i=0, \dots, 2M_t).$$

Differentiating $A' = A$ and $B' = B$ partially with respect to $Y_{\lambda}^{(t)}$ and applying (32), we get

$$(33) \quad 2 \sum_{g, h=1}^M A_{gh} X_g d_{ht\lambda} + \sum_{q=1}^L \sum_{p=1}^{M_t} (X_{p-1}^{(q)} f_{g\mu t\lambda} + X_{\mu}^{(q)} f_{q p-1 t\lambda}) = \begin{cases} Y_{\lambda+1+M_t}^{(t)} & (\lambda=0, \dots, M_t-1), \\ 0 & (\lambda=M_t), \\ Y_{\lambda-1-M_t}^{(t)} & (\lambda=1+M_t, \dots, 2M_t); \end{cases}$$

$$(34) \quad 2 \sum_{g, h} B_{gh} X_g d_{ht\lambda} + \sum \sum (X_p^{(q)} f_{q\mu t\lambda} + X_{\mu}^{(q)} f_{qp t\lambda}) = \begin{cases} 0 & (\lambda=0), \\ Y_{\lambda+M_t}^{(t)} & (\lambda=1, \dots, M_t), \\ Y_{\lambda-M_t}^{(t)} & (\lambda=1+M_t, \dots, 2M_t). \end{cases}$$

In (34₂) we replace λ by $\lambda + 1$ and compare the result with (33₁); by the coefficients of X_p , we get

$$(35) \quad \sum_{h=1}^M B_{gh} d_{ht\lambda+1} = \sum_{h=1}^M A_{gh} d_{ht\lambda} \quad (\lambda=0, \dots, M_t-1; g=1, \dots, M).$$

By the coefficient of X_g in (33₂), we get

$$\sum_{h=1}^M A_{gh} d_{htM_t} = 0 \quad (g=1, \dots, M).$$

Since $|A_{gh}| \neq 0$, each $d_{htM_t} = 0$. Then, by (35) for $\lambda = M_t - 1$, $d_{ht\lambda} = 0$. Similarly, by taking in turn $\lambda = M_t - 2, \dots, \lambda = 0$, we get

$$(36) \quad d_{hj} = 0 \quad (j=0, \dots, M_t; h=1, \dots, M; t=1, \dots, L).$$

Thus (36) are among the necessary conditions for the identity of corresponding

F 's in (33) and (34). When the latter conditions are satisfied, we have unique expressions for the F 's in terms of the X 's, and hence the form of the inverse of (32). By (34₂), $F_k^{(i)}$ ($k = 1 + M_i, \dots, 2M_i$) does not contain an X_g , in view of (36). Since the pairs (31) and (31') are of like type, the inverse (32) must have the analogous property that $X_k^{(i)}$ ($k = 1 + M_i, \dots, 2M_i$) shall not contain a variable F_g . But every term of the double sums in (31) contains such an $X_k^{(i)}$. Hence the first sums in (31) must be transformable into the first sums in (31') by a transformation on the X_g alone. We may therefore state the

THEOREM. *The necessary and sufficient condition for the equivalence in F of the singular pairs (31) and (31') is the equivalence in F of the non-singular components*

$$\sum A_{gh} X_g X_h, \quad \sum B_{gh} X_g X_h \quad \text{and} \quad \sum A'_{gh} F_g F_h, \quad \sum B'_{gh} F_g F_h.$$

THE UNIVERSITY OF CHICAGO,

April 4, 1909.
